

# Two-Grid Algorithms for Singularly Perturbed Reaction-Diffusion Problems on Layer Adapted Meshes

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## Abstract

We propose a new two-grid approach based on Bellman-Kalaba quasilinearization [6] and Axelsson [4]-Xu [30] finite element two-grid method for the solution of singularly perturbed reaction-diffusion equations. The algorithms involve solving one inexpensive problem on coarse grid and solving on fine grid one linear problem obtained by quasilinearization of the differential equation about an interpolant of the computed solution on the coarse grid. Different meshes (of Bakhvalov, Shishkin and Vulanović types) are examined. All the schemes are uniformly convergent with respect to the small parameter. We show theoretically and numerically that the global error of the two-grid method is the same as of the nonlinear problem solved directly on the fine layer-adapted mesh.

## 1 INTRODUCTION

We consider the problem

$$\begin{aligned} -\varepsilon^2 u'' + f(x, u) &= 0, \quad x \in \Omega \equiv (0, 1), \\ u(0) &= u(1) = 0, \end{aligned} \quad (1)$$

where  $\varepsilon$  is a small perturbation parameter,  $0 < \varepsilon \ll 1$ . We assume (see Lemma 2 and Theorems 4, 6) that the function  $f$  has the continuous derivatives:

$$\frac{\partial^{i+j} f(x, u)}{\partial x^i \partial u^j}, \quad 0 \leq i+j \leq 4, \quad 0 \leq i \leq 3, \quad 0 \leq j \leq 4, \quad (x, u) \in (\overline{\Omega} \times R), \quad (2)$$

and

$$f_u(x, u) \geq c_0^2 > 0, \quad (x, u) \in (\overline{\Omega} \times R). \quad (3)$$

The condition (3) is the standard stability condition, which implies that both (1) and the reduced problem  $f(x, u) = 0$  have unique smooth solutions  $u_\varepsilon \in C^4(\overline{\Omega})$  and  $u_0$ , respectively.

It is shown theoretically and experimentally in [7] that there exists no finite difference scheme (or finite element approximation) of (1) on standard meshes whose solution can be guaranteed to converge to the solution  $u$  in the maximum norm, uniformly with respect to the perturbation parameter  $\varepsilon$ .

Nowadays, two basic types of non-equidistant (layer adapted) meshes, suggested by Bakhvalov and Shishkin, are used for solving singularly perturbed problems [17, 19]. An

explicit mesh construction method to solve a singularly perturbed problem of type (1) was used first by Bakhvalov [5], where he obtained the special discretization mesh  $w_h = \{x_i = \lambda(i/n) : i = 0, 1, \dots, n\}$ ,  $h = 1/n$ , where by  $\lambda$  is the mesh generating function that consists of three parts:  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$ . The functions  $\lambda_1$  and  $\lambda_3$  generate the mesh points in the boundary layers in the neighborhood of  $x = 0$  and  $x = 1$  respectively. The function  $\lambda_2$  generates the mesh points outside the boundary layers and it is a tangent line to both  $\lambda_1$  and  $\lambda_3$ , and  $\lambda_2(0.5) = 0.5$ .

A much simpler mesh was constructed by Shishkin, see [17, 19], but many difference schemes applied on Bakhvalov's mesh show better results.

Not only to simplify Bakhvalov's mesh but also to increase the density of mesh points in the boundary layers, Vulanović modifies the previously known mesh generating functions [23], see also [8, 16, 19, 25, 26].

There is a wide range of publications that deal with layer adapted meshes. In the monograph [16] an extensive review is given.

Currently there is considerable interest in the construction of high-order approximations to singularly perturbed problems, [16, 19, 21, 22, 24, 25, 26]. But such constructions often lead to an extension of the stencil or to discretizations that are not inverse monotone [16]. Another way to increase the accuracy of the numerical solution to singularly perturbed problems is the use of Richardson extrapolation [16, 19, 21]. However the Richardson procedure requires solution of systems of nonlinear algebraic equations on each of the nested meshes [21].

The main objective of this paper is to present two-grid algorithms using standard difference approximation on different adaptive meshes for the boundary value problem (1). The two-grid method used for high-order and time-effective computations was first introduced by Axellson [4] and Xu [30] independently from each other. It was further investigated by many other authors and for many problems for instance, for elliptic, parabolic and Stokes-Darcy equations, see [18] and the references therein. Note that the error estimates in these papers are in *weak (Sobolev-type)* discrete norms. In comparison, our error estimates below are in the *maximum* norm. This norm is sufficiently strong to capture layers and hence seems most appropriate for singularly perturbed problems.

The rest of the paper is organized as follows. In Section 2 we introduce three meshes. In Section 3 we describe a Newton-Bellman & Kalaba linearization process [6, 15] for the continuous problem (1)-(3) in order not only to prove uniform convergence of the difference scheme but mainly to obtain the estimate (15) which plays a key role in the analysis of the two-grid algorithms in the next section. In Section 4 we describe the two-grid algorithms (TGAs) and provide error estimates for the difference scheme discretization of the TGAs, Theorem 6. Section 5 includes numerical results that illustrate the theoretical estimates. Finally, conclusions and directions for future work are presented.

Although our theoretical results will be presented in a model, one-dimensional classical situation, the algorithm possesses a wider generality, see test Example 2 in Section 5. Also, there are many interesting and relevant boundary value problems of type (1), for which the condition (3) is not satisfied, see [20, 14], and for which our computational techniques still work well.

**Notation** We define a norm of a continuous function  $f(x)$  as  $\|f\| = \max_{x \in \Omega} |f(x)|$ . Let  $w_h = \{0 < x_1 < \dots < x_{n-1} < 1\}$ ,  $x_0 = 0$ ,  $x_n = 1$ ,  $\bar{w}_h = w_h \cup \{x_0\} \cup \{x_n\}$  and  $h_i = x_i - x_{i-1}$ ,  $\bar{h}_i = 0.5(h_i + h_{i+1})$ . For a mesh function  $y$ , we introduce the standard finite difference approximations to the first and second derivatives, [17, 19]:

$$y_{\bar{x},i} = (y_i - y_{i-1})/h_i, \quad y_{x,i} = y_{\bar{x},i+1}, \quad y_{\widehat{x},i} = (y_i - y_{i-1})/\bar{h}_i,$$

$$y_{\widehat{xx},i} = \frac{1}{\bar{h}_i} \left( \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right).$$

The discrete maximum norm is defined by  $\|y\| = \|y\|_h = \max_{0 \leq i \leq n} |y_i|$ . Throughout this paper  $C$  and  $C_i$ ,  $i \geq 0$ , denote positive constants independent on  $N$  (the number of coarse mesh nodes, respective mesh step  $H$ ),  $n$  (the number of fine mesh nodes, respective mesh step  $h$ ) and  $\varepsilon$ .

## 2 THE LINEAR PROBLEM ON ADAPTIVE MESHES

For a given integer  $n$ , on  $\overline{\Omega} = [0, 1]$  we introduce the special mesh  $\overline{w}_h$  with  $x_i = \lambda(i/n)$ ,  $i = 0, 1, \dots, n$ . Following [8, 16, 19] we present all such meshes by their mesh-generating functions. Bakhvalov's mesh-generating function is given by

$$\lambda(t) = \begin{cases} \phi(t) := a\varepsilon \ln \frac{q}{q-t}, & t \in [0, \alpha], \\ \phi(\alpha) + \phi'(\alpha)(t - \alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1], \end{cases}$$

where  $a$  and  $q$  are constants, independent of  $\varepsilon$ , such that

$$q \in (0, 0.5), \quad a \in \left(0, \frac{q}{\varepsilon}\right). \quad (4)$$

Here  $\alpha$  is the abscissa of the contact point of the tangent line from  $(0.5, 0.5)$  to  $\phi(t)$ . The generated mesh will be called  $B$  - mesh.

Shishkin's mesh (S-mesh) is a piecewise equidistant and consequently much simpler than the mesh above. The generating function for this mesh is

$$\lambda(t) = \begin{cases} 4\alpha t, & t \in [0, \alpha], \\ \alpha + 2(1 - 2\alpha)(t - 0.25), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1] \end{cases}$$

with

$$\alpha = \min\{1/4, 2\gamma_0^{-1}\varepsilon \ln N\}, \quad \gamma_0 = \min\{c_0, 1\}.$$

Vulanović, see [8, 23, 25], has shown that  $\lambda$  does not need to be a logarithmic function. A class of suitable mesh generating functions was given and it includes functions of a much simpler rational form. From those functions we select the following one

$$\lambda(t) = \begin{cases} \mu(t) := \frac{a\varepsilon t}{q-t}, & t \in [0, \alpha], \\ \mu(\alpha) + \mu'(\alpha)(t - \alpha), & t \in [\alpha, 0.5], \\ 1 - \lambda(1 - t), & t \in [0.5, 1], \end{cases}$$

where  $q$  and  $a$  satisfy the conditions (4) and the parameter  $\alpha$  has the same meaning as in the Bakhvalov mesh, but it can be explicitly calculated,

$$\alpha = \frac{q - \sqrt{\varepsilon a q(1 - 2q + 2\varepsilon a)}}{1 + 2\varepsilon a}.$$

This mesh will be called V-mesh.

In the next section we shall develop a linearization procedure for the solution of problem (1)-(3). At each iteration we solve linear two-point boundary value problems of the following type:

$$\begin{aligned} -\varepsilon^2 u'' + b(x, \varepsilon)u &= g(x, \varepsilon), \quad x \in \Omega, \\ u(0) &= 0, \quad u(1) = 0. \end{aligned} \quad (5)$$

The functions  $b(x, \varepsilon)$  and  $g(x, \varepsilon)$  are assumed to be in  $C^3(\overline{\Omega})$  for a fixed parameter  $\varepsilon \in [0, 1]$ . On each of the meshes above we shall consider the classical three-point difference scheme for the problem (5):

$$\begin{aligned} -\varepsilon^2 y_{\overline{x},i} + b(x_i, \varepsilon) y_i &= g(x_i, \varepsilon), \\ y_0 &= 0, \quad y_n = 0. \end{aligned} \quad (6)$$

Using the discrete Green's function method, optimal convergence results for the discrete solutions of two-point boundary value problems can be obtained, see [1, 20, 16, 19] and references therein. On the base of classical results for the case  $b(x, \varepsilon) = b(x)$ ,  $g(x, \varepsilon) = g(x)$ , see [5, 17, 19, 20] one can prove easily the following theorem:

**Theorem 1** *Let  $b, g$  have continuous derivatives with respect to  $x$  up to order three that are uniformly bounded with respect to  $\varepsilon \in [0, 1]$  and  $b(x, \varepsilon) \geq \beta > 0$  for all  $(x, \varepsilon) \in (\overline{\Omega} \times [0, 1])$ . If  $u$  is the solution of problem (5) and  $y$  of problem (6), then for the error of the difference scheme (6) the following estimate holds:*

$$\|u - y\| \leq C n^{-2} \ln^k n, \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}. \end{cases} \quad (7)$$

### 3 UNIFORM CONVERGENCE VIA NEWTON'S LINEARIZATION

On the mesh  $\overline{w}_h$  the scheme (6) for (1)-(3) is defined as follows:

$$\begin{aligned} -\varepsilon^2 y_{\overline{x},i} + f(x_i, y_i) &= 0, \quad i = 1, \dots, n-1, \\ y_0 &= 0, \quad y_n = 0. \end{aligned} \quad (8)$$

The uniform convergence for this scheme was analyzed in many papers [19, 23, 24]. We shall also address this using a linearization Newton-Bellman&Kallaba procedure [6, 15]. In a natural way this will lead us to the idea of the two-grid method.

The following assertion is well known and often used to prove uniform convergence of the scheme (8) (see for example [19, 24, 25]).

**Lemma 2** *Let the conditions (2), (3) be satisfied. Then:*

- a) *the problem (1)-(3) has unique solution  $u \in C^4(\overline{\Omega})$ .*
- b) *the solution  $u$  satisfies the estimates:*

$$\|u^{(j)}\| \leq C(1 + \varepsilon^{-j}(\exp(-c_0 x/\varepsilon) + \exp(-c_0(1-x)/\varepsilon)))$$

for  $j = 1, 2, 3$ . The estimate for  $j = 0$  looks as follows:

$$\|u\| \leq c_0^{-2} \|f(x, 0)\|.$$

We use the quasilinearization of Bellman&Kalaba for studying the problem (1)-(3):

$$\begin{aligned} L^m u^{(m+1)} &\equiv -\varepsilon^2 \frac{d^2 u^{(m+1)}}{dx^2} + f'_u(x, u^{(m)}) u^{(m+1)} = -f_u(x, u^{(m)}) + f'_u(x, u^{(m)}) u^{(m)} \\ u^{(m+1)}(0) &= 0, \quad u^{(m+1)}(1) = 0, \quad m = 0, 1, 2, \dots \end{aligned} \quad (9)$$

Let us first establish convergence of the linearization process. Suppose that

$$\|u - u^{(0)}\| \leq \rho. \quad (10)$$

Let

$$\theta = \max_{x \in \overline{\Omega}, |\xi| \leq l+2\rho} \|f''_{uu}(x, \xi)\|.$$

**Lemma 3** Assume that  $c_0^{-2}\theta\rho < 1$ . Then

$$\|u^{(m)} - u\| \leq c_0^{-2}\theta\rho^{2^m}, \quad m = 0, 1, 2, \dots \bullet \quad (11)$$

Also, if the function  $u^0(x)$  is in  $C^3$  and satisfies estimates of type b) for  $j = 0, 1$  in Lemma 2, then for the solution  $u^{m+1}$  assertions of type a), b) in Lemma 2 hold.

*Proof.* The boundary value problem for  $v = u^{(m+1)} - u$  reads as follows:

$$L^m v = F^{(m)}(x), \quad v(0) = 0, \quad v(1) = 0, \quad (12)$$

where

$$F^{(m)} = f(x, u^{(m)}) - f(x, u) + f'_u(x, u^{(m)})u^{(m)} - f'_u(x, u)u.$$

We will prove by induction that for all  $s \geq 0$ ,  $\|u^{(s)} - u\| \leq \rho$ . For  $k = 0$  this inequality is obvious. Suppose that it holds for  $s = m$ . Using the mean value theorem, we easily obtain  $\|F^{(m)}\| \leq \theta\|u^{(m)} - u\|^2$ . The maximum principle applied to problem (12) implies:

$$\|u^{(m+1)} - u\| \leq c_0^{-2}\theta\|u^{(m)} - u\|^2. \quad (13)$$

By the assumptions  $\|u^{(m)} - u\| \leq \rho$  and  $c_0^{-2}\theta\rho < 1$  we reach the next step of the induction. So that for all  $m \geq 0$  we have  $\|u^{(m)} - u\| \leq \rho$ . Therefore, (13) holds for all  $m \geq 0$  and this implies (11).  $\square$

Furthermore, after appropriate choice of  $u^0$  ( $u^0 = y_H^{I(x)}$ ) in Algorithm 1, and taking into account of assumptions (2), (3) the equation (9) at  $m = 0$  takes the standard form studied in [17, 19]. Then, by induction one can easily prove the second part in the formulation of the present Lemma.

Let us consider the finite-difference analogue of the iterative process (9):

$$\begin{aligned} L_h y_i^{(m+1)} &= \varepsilon^2 y_{\bar{x}\bar{x},i}^{(m+1)} + f'_u(x_i, y_i^{(m)})y_i^{(m+1)} \\ &= -f(x_i, y_i^{(m)}) + f'_u(x_i, y_i^{(m)})y_i^{(m)}, \quad i = 1, \dots, n-1, \\ y_0^{(m+1)} &= 0, \quad y_N^{(m+1)} = 0, \quad y_i^{(0)} = u^{(0)}(x_i), \quad i = 1, \dots, n-1, \quad m = 0, 1, 2, \dots \bullet \end{aligned} \quad (14)$$

**Theorem 4** Under the conditions (2), (3) there exist constants  $n_0$  and  $\rho_0$ , independent of  $\varepsilon$ , such that if  $n \geq n_0$  and  $\|u^{(0)} - u\| \leq \rho \leq \rho_0$ , then the following estimate holds:

$$\begin{aligned} \|y^{(m)} - u\| &\leq C \left[ n^{-2} \ln^k n + (c_0^{-2}\theta\rho)^{2^m} \right], \\ \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}. \end{cases} \quad m = 0, 1, 2, \dots \bullet \end{aligned} \quad (15)$$

*Proof.* Assuming that  $c_0^{-2}\theta\rho < 1$ , we introduce the auxiliary iterative process:

$$\begin{aligned} -\varepsilon^2 \tilde{y}_{\bar{x}\bar{x},i}^{(m+1)} + f'_u(x_i, u^{(m)}(x_i))\tilde{y}_i^{(m+1)} \\ = -f(x_i, u^{(m)}(x_i)) + f'_u(x_i, u^{(m)}(x_i))u^{(m)}(x_i), \quad i = 1, \dots, n-1, \\ \tilde{y}_0^{(m+1)} = 0, \quad \tilde{y}_n^{(m+1)} = 0, \quad m = 0, 1, 2, \dots \bullet \end{aligned}$$

An application of Theorem 1 provides the estimate

$$\begin{aligned} \|\tilde{y}^{(m+1)} - u^{(m+1)}\| &\leq C n^{-2} \ln^k n, \\ \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}, \end{cases} \quad m = 0, 1, 2, \dots \bullet \end{aligned} \quad (16)$$

Now we will show that  $\|y^{(m+1)} - \tilde{y}^{(m+1)}\|$ .  $v^{(m+1)} = y^{(m+1)} - \tilde{y}^{(m+1)}$  satisfies the difference problem:

$$L_h v_i^{(m+1)} = F_i^{(m)}, \quad i = 1, \dots, n-1, \quad v_0^{(m+1)} = 0, \quad v_n^{(m+1)} = 0, \quad (17)$$

where for  $F_i^{(m)}$  we have the representation

$$\begin{aligned} F_i^{(m)} = & -(y_i^{(m)} - u^{(m)}(x_i))(f_{uu}(x_i, \xi_i^{(4)})(\xi_i^{(1)} - \xi_i^{(3)}) \\ & + \tilde{y}_i^{(m+1)}(f_{uu}(x_i, \xi_i^{(2)})) - f_{uu}(x_i, \xi_i^{(3)}) + f_{uu}(x_i, \xi_i^{(3)})(\tilde{y}_i^{(m+1)} - \xi_i^{(3)})), \end{aligned}$$

and all  $\xi_i^{(j)}$   $j = 1, 2, 3, 4$  lie between  $y_i^{(m)}$  and  $u^{(m)}(x_i)$ . We estimate  $\tilde{y}_i^{(m+1)} - \xi_i^{(3)}$  using

$$\tilde{y}_i^{(m+1)} - \xi_i^{(3)} = (\tilde{y}_i^{(m+1)} - u^{(m+1)}(x_i)) + (u^{(m+1)}(x_i) - u^{(m)}(x_i)) + (u^{(m)}(x_i) - \xi_i^{(3)}).$$

Starting from (17) and using (11), (16), by an analogical way as in Lemma 3, one can show that for sufficiently large  $n_0$  and small  $\rho_0$  the following estimate holds  $\|y^{(m)} - u^{(m)}\| \leq \rho$  for all  $m \geq 0$ . Therefore,  $\xi_i^{(j)}$ ,  $j = 1, 2, 3, 4$  are bounded and there exist constants  $C_3, C_4, C_5$  such that:

$$\|F^{(m)}\| \leq (C_3\|y^{(m)} - u^{(m)}\| + C_4\|u^{(m+1)} - u^{(m)}\| + C_5n^{-2} \ln^k n)\|y^{(m)} - u^{(m)}\|. \quad (18)$$

We used that the continuous functions  $f_{uu}$ ,  $f_{uuu}$  with given arguments are bounded. Applying the maximum principle to problem (17) and using (16), (18), we obtain

$$\begin{aligned} \|y^{(m+1)} - u^{(m+1)}\| & \leq \\ c_0^{-1}(C_3\|y^{(m)} - u^{(m)}\| + C_4\|u^{(m+1)} - u^{(m)}\| + C_5h^2)\|y^{(m)} - u^{(m)}\| + C_2n^{-2} \ln^k n. \end{aligned} \quad (19)$$

We make another restriction on  $n_0$  and  $\rho_0$ :

$$n^{-2} \ln^k n \leq c_0^{-2}/(6C_5), \quad \rho \leq \min(\alpha/(6C_4), \alpha/(6C_3)).$$

Now, in view of  $\|y^{(m)} - u\| \leq \rho$ , it follows from (19), that

$$\|y^{(m+1)} - u^{(m+1)}\| \leq 0.5\|y^{(m)} - u\| + C_2n^{-2} \ln^k n, \quad m \geq 0.$$

Hence,

$$\|y^{(m)} - u^{(m)}\| \leq C_2n^{-2} \ln^k n, \quad m \geq 0.$$

Finally, (11) implies (15).  $\square$

Now, we are in a position to prove that the scheme (8) is uniformly convergent with respect to  $\varepsilon$ .

**Corollary 5** *Let  $u$  be the solution of problem (1)-(3) and  $y$  f the discrete problem (8). Then he following estimate of type (7) holds true*

$$\|u - y\| \leq Cn^{-2} \ln^k n, \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}. \end{cases} \quad (20)$$

*Proof.* Let us chose  $n_0$ ,  $\rho_0$  in agreement with the requirement of Theorem 4:  $n \leq n_0$ ,  $\rho \leq \rho_0$ . Similarly as in Theorem 4 one can prove that

$$\|y^{(m)} - u\| \leq c_0^2 \theta_0^{-1} (c_0^{-2} \theta \|y^{(0)} - y\|)^{2^m}, \quad m = 0, 1, 2, \dots$$

Therefore

$$y^{(m)} \rightarrow y, \quad \text{as } m \rightarrow \infty,$$

if  $\rho = c_0^{-2} \theta \|y^{(0)} - y\| < 1$ . Let  $m \rightarrow \infty$ , then from (15) we get the required estimate.

To complete the proof let us consider the case  $n < n_0$ . The maximum principle implies  $\|y\| \leq l$  ( $l$  corresponds to those in Lemma 1,b). Hence

$$\|y - u\| \leq \|y\| + \|u\| \leq 2l = Cn^{-2} \ln^k n, \quad C = 2l \begin{cases} \frac{n^2}{\ln^2 n} \text{ on } S - \text{mesh}, \\ n^2 \text{ on } B \text{ and } V - \text{meshes}. \end{cases} \quad \square$$

## 4 TWO-GRID ALGORITHMS

In this section we propose two-grid algorithms *based on the estimate (15)*. For this we introduce the fine grid  $\bar{w}_h$  with  $x_i = \lambda(i/n)$ ,  $i = 1, \dots, n$ , and  $n = N^r$ , where  $r > 1$  is a real number that will be chosen later, see Theorem 6 and comments on Table 6 in Section 5.

A nice property of Shishkin (Bakhalov and Vulanović) [8, 19, 29] meshes is parameter uniform interpolation. On the coarse grid

$$w_H = \{0 = X_0 < X_1 < \dots < X_{N-1} < X_N = 1\},$$

define the linear interpolation of the solution of the discrete problem (8)

$$y_N^I(x) = \sum_{i=0}^N y_i \phi_i(x),$$

where  $\phi_i(x)$  is the standard piecewise linear basis function associated with the interval  $[X_{i-1}, X_{i+1}]$ . For the interpolant  $y_N^I(x)$  the following estimate holds

$$\begin{aligned} \|y_N^I(x) - u\| &\leq \|u^I - u\| + \|y_N^I - u^I\| \\ &\leq CN^{-2} \ln^k N, \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}, \end{cases} \end{aligned}$$

where  $u^I = u^I(x)$  is the interpolant of the continuous solution  $u$ . If in the iterative process (9) one takes  $m = 1$  and the initial guess  $u^{(0)}(x) = y_N^I(x)$ , then in (15) we will have

$$(c_0^{-2} \theta \rho)^2 = CN^{-4} \ln^{2k} N, \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}. \end{cases}$$

Then

$$\|y^1 - u\|_h \leq C \left[ N^{-2r} \ln^k N + N^{-4} \ln^{2k} N \right], \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}, \end{cases}$$

Our first algorithm reads as follows.

### Algorithm 1

**Step 1.** Solve the discrete problem (8) on the coarse grid  $w_H$  and then perform a linear interpolation to obtain the function  $y_H^I(x)$  defined in the domain  $\bar{\Omega} = [0, 1]$ .

**Step 2.** Solve the linear discrete problem

$$\begin{aligned} & -\varepsilon^2 y_{\bar{x}\bar{x},i} + f'_u(x_i, y_N^I(x)) y_i \\ & = f'_u(x_i, y_N^I(x)) y_i - f(x_i, y_H^I(x_i)), \quad i = 1, \dots, n-1, \\ & y_0 = 0, \quad y_n = 0 \end{aligned}$$

to find the fine mesh numerical solution  $y^h$ .

**Step 3.** Interpolate  $y^h$  to obtain  $u_{h,H}^I(x)$ ,  $x \in \bar{\Omega}$ .

The next theorem gives the main theoretical result of the present paper.

**Theorem 6** *Let the conditions (2), (3) hold and  $n = N^2$ , i.e.  $r = 2$ . Then the following error estimate holds true:*

$$\|u_{h,H}^I - u\|_H = CN^{-4} \ln^{2k} N, \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}. \end{cases} \quad (21)$$

Table 1: Points in boundary layers (%) of coarse grid (N) and fine grid (n).

	N	8	16	32	64
	n	64	256	1024	4096
S - mesh	Step 1	25	12.5	12.5	6.25
	Step 2	6.25	4.69	3.71	3.03
V - mesh	Step 1	50	50	43.75	40.63
	Step 2	40.63	40.63	40.04	40.04
B - mesh	Step 1	25	25	18.75	18.75
	Step 2	18.75	17.97	17.77	17.72

It is clear that we can repeat Algorithm 1 to obtain, on the fine mesh  $w_h$ , with  $n = N^4$ , the accuracy

$$CN^{-8} \ln^{4k} N, \quad \begin{cases} k = 2 \text{ on } S - \text{mesh}, \\ k = 0 \text{ on } B \text{ and } V - \text{meshes}. \end{cases}$$

#### Algorithm 2

**Step 1.** For  $m = 0$  do step 1 of Algorithm 1.

**Step 2.** For  $m = 1, 2, \dots$  repeat step 2 of Algorithm 1 with final mesh step corresponding to  $n = N^{2^m}$  and  $y_N^I(x) := u_{h^{m-1}, H}^I(x)$ .

The rate of convergence is the same as in Algorithm 1. However, there is a significant decrease in the computational cost.

**Remark 1.** The formulas of two-grid Axelsson-Xu type algorithms [4, 18, 30] involve the second derivative  $f_{uu}$ , while our Algorithms 1, 2 are free of the second derivative.

## 5 NUMERICAL RESULTS

In this section we discuss numerical results for a set of computational experiments associated with the TGAs.

In order to emphasize the difference between meshes we present Table 1, where the percentage of the number of mesh points in the boundary layers, i.e. in  $[0, \varepsilon] \cup [1 - \varepsilon, 1]$ , is given, with  $q = 0.4$ ,  $a = 1$  and  $\varepsilon = 2^{-8}$ .

**Example 1.** We first consider the test problem [24]

$$-\varepsilon^2 u'' + \frac{u-1}{2-u} + f(x) = 0, \quad u(0) = u(1) = 0,$$

where  $f(x)$  is chosen so that the exact solution is

$$u_\varepsilon(x) = 1 - \frac{\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon)}{1 + \exp(-1/\varepsilon)}.$$

The tables below present the errors

$$E_N = \|u_\varepsilon - y\|,$$

where  $y$  is the numerical solution on a mesh with  $N$  mesh steps. Also, we calculated numerical orders of convergence by formula

$$O_N = \frac{\ln E_N - \ln E_{2N}}{\ln 2}.$$

Also, we introduce  $O_n = O_N(TGAs(N^2))$ , i.e. the order of convergence of TGAs with  $r = 2$ , and  $O_{N^r} = O_N(TGA(N^r))$ , the order of convergence of TGAs with  $n = N^r$ .



Table 2: (Example 1) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the two-grid algorithm 1 (TGAs1) on S-mesh.

$\varepsilon$	N	8	16	32	64
	n	64	256	1024	4096
$10^{-1}$	Step 1	3,230E-02	7,500E-3	1,900E-3	4.682E-4
	$O_N$	1,4602	1,9809	2,0209	
	Step 2	8,295E-4	4,450E-5	2,844E-6	1,763E-7
	$O_n$	4,2204	3,9679	4,0117	
$10^{-2}$	Step 1	7,460E-2	3,920E-2	1,480E-2	5,300E-3
	$O_N$	0,9283	1,4053	1,4815	
	Step 2	7,000E-3	1,100E-3	1,299E-4	1,298E-5
	$O_n$	2,6699	3,0823	3,3234	
$10^{-4}$	Step 1t	7,460E-2	3,920E-2	1,480E-2	5,300E-3
	$O_N$	0,9283	1,4053	1,4815	
	Step 2	7,000E-3	1,100E-3	1,299E-4	1,298E-5
	$O_n$	2,6699	3,0821	3,3234	

Numerical results are presented in Tables 2, 3, 4 which validate the theoretical ones established in Theorem 2 and Corollary 5. It is interesting to discuss the computations for small  $\varepsilon$ . In this case the methods are uniformly convergent, the errors stabilize for each  $N$  as  $\varepsilon \rightarrow 0$ . See the results in the Tables for  $\varepsilon = 10^{-2}$ ,  $\varepsilon = 10^{-4}$ . So, we will discuss the correspondent rows in the tables for  $\varepsilon = 10^{-2}$ . For example the maximum error at the Step1 for  $N = 64$  in Table 2 (S-mesh) is  $5.300e - 3$ , in Table 3 (V-mesh) is  $1.500e - 3$  and in Table 4 (B-mesh) is  $8.491e - 4$ , while at the Step2 the corresponding errors are  $1.298e - 5$ ,  $7.363e - 7$ ,  $3.313e - 7$ . Therefore:

- (1) the TGAs significantly increase the accuracy and the experiments confirm Theorem 2 and Corollary 5;
- (2) it is known (see [8]) that the most accurate is the B-mesh and now for the TGAs the situation is similar.

Finally, the CPU time (boldface numbers) is given in Table 4. For example for  $\varepsilon = 10^{-1}$ , one must compare the value 0.1406 with 0.0938, 1.2344 with 0.1875, 34.9688 with 8.8906. The computational cost of the TGAs is significant and decreases with  $N$ .

Table 5 presents results for the two-grid algorithm 2 (TGAs2). One solves the nonlinear problem (1) on the coarse grid ( $N = 4$ ) and after this (1) is linearized about the interpolant of the numerical solution. Then the linearized problem is solved on the fine mesh with  $n = N^2 = 16$ . Next, again the problem (1) is linearized, but about the interpolant of the last numerical solution and the obtained linearized problem is solved on the fine mesh with  $n = N^4 = 256$ . Therefore, this procedure is equivalent to solving the problem (1) on a coarse grid with  $N = 16$  and then the corresponding linearized problem on a fine mesh with  $n = 256$ . The advantage of the TGAs2 is in decreasing of the number of the algebraic equations of the nonlinear difference problem.

**Example 2.** The theoretical results in Sections 3, 4 concern the classical case of singularly perturbed reaction problems. We will demonstrate by this example the efficiency of the TGAs applied to another class of problems. We consider the test problem in [26]:

$$-\varepsilon^2 \left( \frac{u'}{u+1} \right)' + u = f(x), \quad u(0) = 1, u(1) = \beta.$$

The method discussed in [26] is the central finite-difference scheme applied on meshes of Bakhvalov and piecewise-equidistant types. Here we used a similar discretization.

Table 3: (Example 1) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the (TGAs1) on V-mesh,  $a = 1, q = 0.4$ .

$\varepsilon$	N	8	16	32	64
	n	64	256	1024	4096
$10^{-1}$	Step 1	2,790E-2	8,000E-3	2,000E-3	5,177E-4
	$O_N$	1,8022	2,0000	1,9498	
	Step 2	6,929E-4	4,647E-5	2,942E-6	1,873E-7
	$O_n$	3,8985	3,9812	3,9738	
$10^{-2}$	Step 1	1,368E-1	2,090E-2	5,900E-3	1,500E-3
	$O_N$	2,7105	1,8247	1,9758	
	Step 2	4,600E-03	2,051E-4	1,246E-5	7,363E-7
	$O_n$	4,4873	4,0404	4,0813	
$10^{-4}$	Step 1	1,493E-1	2,220E-2	5,900E-3	1,500E-3
	$O_N$	2,7496	1,9118	1,9758	
	Step 2	5,400E-3	2,170E-4	1,248E-5	7,363E-7
	$O_n$	4,6369	4,1207	4,0827	

We chose  $f(x)$  and  $\beta$  so that the exact solution is

$$u_\varepsilon(x) = \exp(-x/\varepsilon) + \exp(x) - 1.$$

Now the solution has a boundary layer at  $x = 0$ . The results in Tables 6, 7, 8 are similar to those of Example 1. In Table 6 we also give experimental results concerning an optimal choice of the number  $r$  based on the approximate (up to constant multiplier) relation  $\frac{N^r}{r} \approx \frac{N^2}{\ln N}$ . The accuracy is lower and the convergence is slower but there is a decrease in the computational cost.

## 6 Conclusions and future work

In this paper, we proposed two-grid algorithms for the finite difference solution of singularly perturbed reaction-diffusion problems. In these two-grid algorithms, the solution of the fully nonlinear *coarse* problem is used in a single-step *linear* fine mesh problem. Numerical experiments demonstrate that the two-grid algorithms are *dramatically more efficient* than the standard one-grid algorithm.

We have studied the semilinear Dirichlet boundary value problem (1)-(3). It is clear that the present approach can be easily extended to equation (1) with any set of linear two-point boundary conditions [2, 3]. Our future work will be devoted to the development of the proposed algorithms for convection-dominated equations and systems of equations in one and two dimensions, for example the models in finance [9, 10, 12] and medicine [11]. Results concerning the two-grid algorithms combined with exponential difference schemes on standard meshes for singularly perturbed nonlinear ordinary differential equations and systems of equations were reported in [27, 28].

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Table 4: (Example 1) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the (TGAs1) on B-mesh,  $a = 4, q = 0.4$ .

$\varepsilon$	N	8	16	32	64	256	1024
	n	64	256	1024	4096	-	-
$10^{-1}$	Step 1	3,230E-2	7,500E-3	1,900E-3	4,682E-4	2,925E-5	1,8281e-6
	$O_N$	2,1066	1,9809	2,0209	2,0005		
	CPU	0.0469	0.0625	0.0781	<b>0.1406</b>	<b>1.2344</b>	<b>34.9688</b>
	Step 2	8,295E-4	4,450E-5	2,844E-6	1,763E-7		
	$O_n$	4,2204	3,9679	4,0117			
	CPU	<b>0.0938</b>	<b>0.1875</b>	<b>0.6094</b>	8.8906		
$10^{-2}$	Step 1	5,810E-2	1,380E-2	3,400E-3	8,491E-4	5,309E-5	3,3181e-6
	$O_N$	2,0739	2,0211	2,0016	1,9994		
	CPU	0.0313	0.0469	0.0938	<b>0.1406</b>	<b>1.2656</b>	<b>34.7500</b>
	Step 2	1,700E-3	8,720E-5	5,323E-6	3,317E-7		
	$O_n$	4,2850	4,0340	4,0044			
	CPU	<b>0.1094</b>	<b>0.1719</b>	<b>0.5781</b>	7.7344		
$10^{-4}$	Step 1	5,810E-2	1,380E-2	3,400E-3	8,491E-4	5,309E-5	3,3181e-6
	$O_N$	2,0739	2,0211	2,0016	1,9994		
	CPU	0.0581	0.0781	0.0938	<b>0.1494</b>	<b>1.2500</b>	<b>34.8594</b>
	Step 2	1,700E-3	8,720E-5	5,323E-6	3,317E-7		
	$O_n$	4,2850	4,0340	4,0044			
	CPU	<b>0.1094</b>	<b>0.2031</b>	<b>0.5625</b>	6.9844		

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Table 5: (Example 1) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the (TGAs2) on V-mesh,  $a = 3, q = 0.4$ .

$\varepsilon$	N	4	8	16
	n	16	64	256
	$n^2$	256	4096	65536
$10^{-1}$	Step 1	1.094E-1	2.780E-2	7.100E-3
	$O_N$	1.9765	1.9692	
	Step 2	2.412E-3	1.555E-4	9.744E-6
	$O_n$	3.9552	3.9963	
	Step 3	6.729E-6	2.759E-8	1.113E-10
	$O_{n^2}$	7.9299	7.9533	
$10^{-2}$	Step 1	2.163E-1	5.480E-2	1.390E-2
	$O_N$	1.9808	1.9791	
	Step 2	5.280E-2	3.500E-3	2.446E-4
	$O_n$	3.9151	3.8389	
	Step 3	1.088E-3	5.094E-6	2.292E-8
	$O_{n^2}$	7.3213	7.7918	
$10^{-4}$	Step 1	4.334E-1	1.103E-1	2.780E-2
	$O_N$	1.9743	1.9883	
	Step 2	3.887E-2	3.502E-3	2.546E-4
	$O_n$	3.4724	3.7819	
	Step 3	8.016E-4	5.225E-6	2.301E-8
	$O_{n^2}$	7.2614	7.8267	

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Table 6: (Example 2) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the (TGAs1) on S-mesh.

$\varepsilon$	N	8	16	32	64
	n	64	256	1024	4096
	r	1.9752	1.8550	1.8131	1.7984
	$N^r$	<b>60</b>	<b>172</b>	<b>536</b>	<b>1772</b>
$10^{-1}$	Step 1	4,424E-03	1,079E-03	2,677E-04	6,678E-05
	$O_N$	2,0109	2,0109	2,0029	
	Step 2	6,248E-04	8,443E-05	1,454E-05	1,472E-06
	$O_n$	2,8877	2,5376	3,3048	
	Step 2	1,550E-03	2,086E-04	2,575E-05	3,010E-06
	$O_{N^r}$	<b>2,8931</b>	<b>3,0180</b>	<b>3,0968</b>	
$10^{-2}$	Step 1	8,790E-03	3,834E-03	1,477E-03	5,255E-04
	$O_N$	1,1970	1,3762	1,4909	
	Step 2	3,001E-03	8,396E-04	1,582E-04	2,094E-05
	$O_n$	1,8374	2,4079	2,9175	
	Step 2	5,322E-03	1,891E-03	4,704E-04	9,902E-05
	$O_{N^r}$	<b>1,4309</b>	<b>2,0071</b>	<b>2,2482</b>	
$10^{-4}$	Step 1	3.128E-2	1.085E-2	2.683E-3	6.695E-4
	$O_N$	1.5279	2.0153	2.0031	
	Step 2	1.935E-3	1.168E-4	7.087E-6	4.444E-7
	$O_n$	4.0505	4.0427	3.9953	
	Step 2	5,293E-03	1,888E-03	4,708E-04	9,912E-05
	$O_{N^r}$	<b>1,4871</b>	<b>2,0037</b>	<b>2,2479</b>	

Table 7: (Example 2) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the (TGAs1) on V-mesh,  $a = 2, q = 0.4$ .

$\varepsilon$	N	8	16	32	64
	n	64	256	1024	4096
$10^{-1}$	Step 1	8,117E-3	1,835E-3	5.333e-4	1.331e-4
	$O_N$	2.1699	1.7550	2.0020	
	Step 2	4.470e-4	8.554e-5	6.366e-6	4.387e-7
	$O_n$	2.3855	3.7481	3.8589	
$10^{-2}$	Step 1	3.079E-2	1.098E-2	2.718E-3	6.775E-4
	$O_N$	1.4873	2.0150	2.0040	
	Step 2	2.056E-3	1.148E-4	6.971E-6	4.368E-7
	$O_n$	4.1623	4.0422	3.9964	
$10^{-4}$	Step 1	3.128E-2	1.085E-2	2.683E-3	6.695E-4
	$O_N$	1.5279	2.0153	2.0031	
	Step 2	1.935E-3	1.168E-4	7.087E-6	4.444E-7
	$O_n$	4.0505	4.0427	3.9953	

Table 8: (Example 2) The maximum error and the numerical orders of convergence for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-4}$  for the scheme (8) and the (TGAs1) on B-mesh,  $a = 2, q = 0.4$ .

$\varepsilon$	N	8	16	32	64
	n	64	256	1024	4096
$10^{-1}$	Step 1	8,668E-3	2,197E-3	5,593E-4	1,413E-4
	$O_N$	1,9802	1,9737	1,9850	
	Step 2	2,627E-4	1,776E-5	1,115E-6	7,353E-8
	$O_n$	3,8863	3,9944	3,9220	
$10^{-2}$	Step 1	2,605E-2	5,828E-3	1,218E-3	2,987E-4
	$O_N$	2.1602	2.2587	2.0274	
	Step 2	5.069E-4	3.051E-5	1.677E-6	1.032E-7
	$O_n$	4.0544	4.1856	4.0216	
$10^{-4}$	Step 1	4.038E-2	1.077E-2	2.666E-3	6.625E-4
	$O_N$	1.9065	2.0146	2.0528	
	Step 2	8.857E-4	6.401E-5	4.009E-6	2.401E-7
	$O_n$	3.7905	3.9970	4.0611	

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